On the Rigidity Theorem for Spacetimes with a Stationary Event Horizon or a Compact Cauchy Horizon

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Abstract

We consider smooth electrovac spacetimes which represent either (A) an asymptotically flat, stationary black hole or (B) a cosmological spacetime with a compact Cauchy horizon ruled by closed null geodesics. The black hole event horizon or, respectively, the compact Cauchy horizon of these spacetimes is assumed to be a smooth null hypersurface which is non-degenerate in the sense that its null geodesic generators are geodesically incomplete in one direction. In both cases, it is shown that there exists a Killing vector field in a one-sided neighborhood of the horizon which is normal to the horizon. We thereby generalize theorems of Hawking (for case (A)) and Isenberg and Moncrief (for case (B)) to the non-analytic case.

1 Introduction

A key result in the theory of black holes is a theorem of Hawking [1, 2] (see also [3, 4]), which asserts that, under certain hypotheses, the event horizon of a stationary, electrovac black hole is necessarily a Killing horizon, i.e., the spacetime must possess a Killing field (possibly distinct from the stationary Killing field) which is normal to the event horizon. The validity of this theorem is of crucial importance in the classification of stationary black holes, since it reduces the problem to the cases covered by the well-known uniqueness theorems for electrovac black holes in general relativity [5, 6, 7, 8, 9, 10]. However, an important, restrictive hypothesis in Hawking's theorem is that the spacetime be analytic.

A seemingly unrelated theorem of Isenberg and Moncrief [11, 12] establishes that in an electrovac spacetime possessing a compact Cauchy horizon ruled by closed null geodesics, there must exist a

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Killing vector field which is normal to the the Cauchy horizon. This result supports the validity of the strong cosmic censorship hypothesis [13] by demonstrating that the presence of such a compact Cauchy horizon is "non-generic". The Isenberg-Moncrief theorem also contains the important, restrictive hypothesis that the spacetime be analytic.

The main purpose of this paper is to show that the theorems of Hawking and of Isenberg and Moncrief can be proven in the case of a smooth (as opposed to analytic) geometrical setting.[1] However, a fundamental limitation of our method is that we are able to prove existence of the Killing field only on a one-sided neighborhood of the relevant horizon. For the Hawking theorem, this one-sided neighborhood corresponds to the interior of the black hole, whereas the existence of a Killing field in the exterior region is what is relevant for the black hole uniqueness theorems. However, for the Isenberg-Moncrief theorem, the one-sided neighborhood corresponds to the original Cauchy development, so our results significantly strengthen their conclusion that the presence of a compact Cauchy horizon ruled by closed null geodesics is an artifact of a spacetime symmetry.

This paper is organized as follows: In Section 2 we consider stationary black hole spacetimes and establish the existence of a suitable discrete isometry which maps each generator of the event horizon into itself. As seen in Section 3, by factoring the spacetime by this isometry, we produce a spacetime having the local geometrical properties of the spacetimes considered by Isenberg and Moncrief. This construction explicitly demonstrates the close mathematical relationship between the Hawking and Isenberg-Moncrief theorems. In Section 3, we also review the relevant result of Isenberg and Moncrief, which shows that in suitably chosen Gaussian null coordinates defined in the "unwrapping" of certain local neighborhoods covering the horizon, $\mathcal{N}$, all the fields and their coordinate derivatives transverse to $\mathcal{N}$ are independent of the coordinate u on $\mathcal{N}$. In the analytic case, this establishes that $(\partial/\partial u)^a$ is (locally) a Killing field. Section 4 contains the key new idea of the paper: We use the methods of [15], [16] to extend the region covered by the local Gaussian null coordinates of Isenberg and Moncrief so that the extended spacetime is smooth and possesses a bifurcate null surface. This bifurcate null surface then provides a suitable initial data surface, from which the existence of a Killing field on the extended (and, hence, on the original) spacetime can be established without appealing to analyticity. The results concerning the null initial value formulation that are needed to establish the existence of a Killing field are proven in an appendix.

Throughout this paper a spacetime (M, g_{ab}) is taken to be a smooth, paracompact, connected, orientable manifold M endowed with a smooth Lorentzian metric g_{ab} of signature $(+,-,-,-)$. It is assumed that (M, g_{ab}) is time orientable and that a time orientation has been chosen. The Latin indices $a, b, c...$ will be used as abstract tensor indices [17], the Latin indices $i, j, k...$ will denote tetrad components (used only in Appendix B), and Greek indices will denote coordinate components.

2 Stationary black hole spacetimes

In this section we shall give a mathematically precise specification of the class of stationary, black hole spacetimes to be considered, and we then shall prove existence of a discrete isometry which maps each generator of the horizon into itself.

We consider smooth, strongly causal spacetimes (M, g_{ab}) which are (k, α)-asymptotically stationary as specified in definition 2.1 of [18]. Thus, we assume that (M, g_{ab}) possesses a one-parameter group of isometries, ϕ_t, generated by a Killing vector field t^a, and possesses a smooth acausal slice Σ which contains an asymptotically flat "end", Σ_{end}, on which t^a is timelike and the properties specified in definition 2.1 of [18] hold. However, the precise asymptotic flatness conditions given in that definition will not be of great importance here, and could be significantly weakened or modified.

[1] Further generalizations to allow for the presence of other types of matter fields will be treated by one of us elsewhere [14].

We further require that if any matter fields (such as an electromagnetic field, F_{ab}) are present in the spacetime, then they also are invariant under the action of ϕ_t.

We define M_{end} to be the orbit[2] of Σ_{end} under the isometries

$$M_{end} = \phi\{\Sigma_{end}\}. \tag{2.1}$$

The black hole region $\mathcal{B}$ is defined to be the complement of $I^-[M_{end}]$ and the white hole region $\mathcal{W}$ is defined to be the complement of $I^+[M_{end}]$. We require that (M, g_{ab}) possess a black hole but no white hole, i.e. $\mathcal{W} = \emptyset$ which implies that

$$M = I^+[M_{end}]. \tag{2.2}$$

Note that the domain of outer communications $\mathcal{D}$ associated with the asymptotically flat end is, in general, defined to be the intersection of the chronological future and past of M_{end}, but, in view of eq.(2.2), we have simply

$$\mathcal{D} = I^-[M_{end}]. \tag{2.3}$$

The (future) event horizon of the spacetime is defined by

$$\mathcal{N} = \partial I^-[M_{end}]. \tag{2.4}$$

Our final requirement is that $\mathcal{N}$ is smooth and that the manifold of null geodesic generators of $\mathcal{N}$ has topology S^2 (so that $\mathcal{N}$ has topology $\mathbb{R} \times S^2$).

Definition 2.1 *Stationary black hole spacetimes which satisfy all of the above assumptions will be referred as spacetimes of class A.*

Remark 2.1 *If it is merely assumed that $\mathcal{N}$ has topology $\mathbb{R} \times K$, where K is compact, then under some mild additional assumptions, it follows from the topological censorship theorem [19] that each connected component of $\mathcal{N}$ has topology $\mathbb{R} \times S^2$ [20, 21, 22]; see remark 2.2 below for a strengthening of this result. However, rather than introduce any additional assumptions here, we have chosen to merely assume that $\mathcal{N}$ has topology $\mathbb{R} \times S^2$.*

We begin with the following lemma

Lemma 2.1 *Let (M, g_{ab}) be a spacetime of class A. Then for all $q \in \mathcal{N}$ and all $t \neq 0$ we have $\phi_t(q) \neq q$. In particular, t^a is everywhere non-vanishing on $\mathcal{N}$.*

Proof Suppose that for some $q \in \mathcal{N}$ and some $t \neq 0$ we had $\phi_t(q) = q$. Since $M = I^+[M_{end}]$, there exists $p \in M_{end}$ such that $p \in I^-(q)$. Since $\phi_{nt}(q) = q$ for all integers n, it follows that $\phi_{nt}(p) \in I^-(q)$ for all n, from which it follows that $\phi\{p\} \in I^-(q)$. Therefore, by lemma 3.1 of [18], we have $I^-(q) \supset M_{end}$ and, hence, $I^-(q) \supset I^-[M_{end}] = \mathcal{D}$. However, since $\mathcal{N} = \partial I^-[M_{end}]$, it follows that q lies on a future inextendible null geodesic, γ, contained within $\mathcal{N}$. Let r lie to the future of q along γ. Let O be an open neighborhood of r which does not contain q and let V be any open neighborhood of r with $V \subset O$. Since $r \in \mathcal{N} = \partial I^-[M_{end}] = \partial\mathcal{D}$, we have $V \cap \mathcal{D} \neq \emptyset$. Hence we can find a causal curve which starts in $V \cap \mathcal{D}$, goes to q (since $I^-(q) \supset \mathcal{D}$) and then returns to r along γ. This violates strong causality at r. $\square$

[2]The orbit of an arbitrary subset $Q \subset M$ under the action of ϕ_t is defined to be $\phi\{Q\} = \cup_{t\in\mathbb{R}}\phi_t[Q]$.

Remark 2.2 *No assumptions about the topology or smoothness of $\mathcal{N}$ were used in the proof of this lemma. It is worth noting that a step in the proof of this lemma can be used to strengthen the results of [20], so as to eliminate the need for assuming existence of an asymptotically flat slice that intersects the null geodesic generators of $\mathcal{N}$ in a cross section. First, we note that part (1) of lemma 2 of [20] can be strengthened to conclude that for any $p \in M_{end}$, each Killing orbit, α, on $\mathcal{N}$ intersects the achronal C^{1-} hypersurface $\mathcal{C} \equiv \partial I^+(p)$ in precisely one point. (Lemma 2 of [20] proved the analogous result for Killing orbits in $\mathcal{D}$.) Namely by Lemma 3.1 of [18], α satisfies either $I^-[\alpha] \cap M_{end} = \emptyset$ or $I^-[\alpha] \supset M_{end}$. The first possibility is excluded by our assumption that $M = I^+[M_{end}]$, so there exists $q \in \alpha \cap I^+(p)$. On the other hand, the proof of the above lemma shows that $I^-(q)$ cannot contain $\mathcal{D}$, so there exists $t > 0$ such that $q \notin I^+(\phi_t(p))$. Equivalently, we have $\phi_{-t}(q) \notin I^+(p)$, which implies that the Killing orbit α must intersect $\mathcal{C}$. Furthermore, if α intersected $\mathcal{C}$ in more than one point there would exist $t > 0$ so that both $r \in \alpha$, and $\phi_t(r)$ lie on $\mathcal{C}$. This would imply, in turn, that r lies on the boundary of both $I^+(p)$ and $I^+(\phi_{-t}(p))$ which is impossible since $p \in I^+(\phi_{-t}(p))$. Consequently, each Killing orbit on $\mathcal{N}$ intersects $\mathcal{C}$ precisely once, i.e., $\varsigma \equiv \mathcal{C} \cap \mathcal{N}$ is a cross-section for the Killing orbits, as we desired to show.[3] In particular, this shows that $\mathcal{N}$ has the topology $\mathbb{R} \times \varsigma$. If we now assume, as in [20], that $\mathcal{D}$ is globally hyperbolic, that the null energy condition holds, and that $[\mathcal{C} \setminus \mathcal{C}_{ext}] \cap \mathcal{D}$ has compact closure in M, then the same argument as used in the proof of Theorem 3 of [20] establishes that each connected component of ς has topology S^2, without the need to assume the existence of an achronal slice which intersects the null geodesic generators of $\mathcal{N}$ in a cross section.*

Our main result of this section is the following.[4]

Proposition 2.1 *Let (M, g_{ab}) be a spacetime of class A which satisfies the null energy condition, $R_{ab}k^a k^b \geq 0$ for all null k^a. Then there exists a $t_0 \neq 0$ such that ϕ_{t_0} maps each null geodesic generator of $\mathcal{N}$ into itself. Thus, the Killing orbits on $\mathcal{N}$ repeatedly intersect the same generators with period t_0.*

Proof By Proposition 9.3.1 of [2], the expansion and shear of the null geodesic generators of $\mathcal{N}$ must vanish. By Lemma B.1 of Appendix B, this implies that $\mathcal{L}_k \, {g'}_{ab} = 0$ on $\mathcal{N}$, where ${g'}_{ab}$ denotes the pullback of g_{ab} to $\mathcal{N}$ and k^a is any smooth vector field normal to $\mathcal{N}$ (i.e., tangent to the null geodesic generators of $\mathcal{N}$). Since we also have ${g'}_{ab}k^b = 0$, it follows from the Appendix of [23] that ${g'}_{ab}$ gives rise to a negative definite metric, $\hat{g}_{AB}$, on the manifold, $\mathcal{S}$, of null geodesic orbits of $\mathcal{N}$. By our assumptions, $\mathcal{S}$ has topology S^2.

Now, for all t, ϕ_t maps $\mathcal{N}$ into itself and also maps null geodesics into null geodesics. Consequently, ϕ_t gives rise to a one parameter group of diffeomorphisms $\hat{\phi}_t$ on $\mathcal{S}$, which are easily seen to be isometries of $\hat{g}_{AB}$. Let $\hat{t}^A$ denote the corresponding Killing field on $\mathcal{S}$. If $\hat{t}^A$ vanishes identically on $\mathcal{S}$ (corresponding to the case where t^a is normal to $\mathcal{N}$), then the conclusion of the Proposition holds for all $t_0 \neq 0$. On the other hand, if $\hat{t}^A$ does not vanish identically, then since the Euler

[3] Furthermore, if $\mathcal{N}$ is smooth and ς is compact, then ς also is a cross-section for the null geodesic generators of $\mathcal{N}$. Namely, smoothness of $\mathcal{N}$ precludes the possibility that a null geodesic generator, γ, of $\mathcal{N}$ has endpoints. (Future endpoints are excluded in any case, since $\mathcal{N}$ is a past boundary.) To show that γ must intersect ς, let $r \in \gamma$ and let t be such that $\phi_{-t}(r) \in \varsigma$. (Such a t exists since ς is a cross section for Killing orbits.) Suppose that $t > 0$. If γ failed to intersect ς, then the segment of γ to the past of r would be a past inextendible null geodesic which is confined to the compact region bounded by ς and $\phi_t[\varsigma]$. This violates strong causality. Similar arguments apply for the case where $t < 0$, thus establishing that γ must intersect ς. Finally, if γ intersected ς at two points, q, s, then by achronality of $\mathcal{C}$, the segment of γ between q and s must coincide with a null geodesic generator, λ, of $\mathcal{C}$. When extended maximally into the past, this geodesic must remain in $\mathcal{N}$ (by smoothness of $\mathcal{N}$) and in $\mathcal{C}$ (since $\mathcal{C}$ is a future boundary and $p \notin \mathcal{N}$). Thus, we obtain a past inextendible null geodesic which lies in the compact set $\varsigma = \mathcal{N} \cap \mathcal{C}$, in violation of strong causality.

[4] Note that the conclusion of this Proposition was assumed to be satisfied in the proof of Prop. 9.3.6 of [2], but no justification for it was provided there.

characteristic of $\mathcal{S}$ is non-vanishing, there exists a $p \in \mathcal{S}$ such that $\hat{t}^A(p) = 0$. By the argument given on pages 119-120 of [24], it follows that there exists a $t_0 \neq 0$ such that $\hat{\phi}_{t_0}$ is the identity map on $\mathcal{S}$. Consequently, ϕ_{t_0} maps each null geodesic generator of $\mathcal{N}$ into itself. □

3 Isenberg-Moncrief Spacetimes

In this Section, we shall consider spacetimes, (M, g_{ab}), which contain a compact, orientable, smooth null hypersurface, $\mathcal{N}$, which is generated by closed null geodesics.

Definition 3.1 *Spacetimes which satisfy the above properties will be referred to as spacetimes of class B.*

Spacetimes of class B arise in the cosmological context. In particular, the Taub-NUT spacetime and its generalizations given in Refs. [25, 26, 27]) provide examples of spacetimes of class B. In these examples, $\mathcal{N}$ is a Cauchy horizon which separates a globally hyperbolic region from a region which contains closed timelike curves. However, it should be noted that even among the Kerr-Taub-NUT spacetimes one can find (see Ref. [25]) spacetimes with compact Cauchy horizons for which almost all of the generators of the horizon are not closed. Therefore it should be emphasized that here we restrict consideration to horizons foliated by circles.

Since strong causality is violated in all spacetimes of class B, it is obvious that no spacetime of class A can be a spacetime of class B. Nevertheless, the following Proposition shows that there is a very close relationship between spacetimes of class A and spacetimes of class B:

Proposition 3.1 *Let (M, g_{ab}) be a spacetime of class A. Then there exists an open neighborhood, $\mathcal{O}$, of the horizon, $\mathcal{N}$, such that $(\mathcal{O}, g_{ab})$ is a covering space of a spacetime of class B.*

Proof Let $t_0 > 0$ be as in Proposition 2.1. By Lemma 2.1, ϕ_{t_0} has no fixed points on $\mathcal{N}$. Since the fixed points of an isometry comprise a closed set, there exists an open neighborhood, $\mathcal{U}$, of $\mathcal{N}$ which contains no fixed points of ϕ_{t_0}. Let $\mathcal{O} = \phi\{\mathcal{U}\}$. Then clearly $\mathcal{O}$ also is an open neighborhood of $\mathcal{N}$ which contains no fixed points of ϕ_{t_0}. Moreover, ϕ_{t_0} maps $\mathcal{O}$ into itself. Let $(\tilde{M}, \tilde{g}_{ab})$ be the factor space of $(\mathcal{O}, g_{ab})$ under the action of the isometry ϕ_{t_0}. Then $(\tilde{M}, \tilde{g}_{ab})$ is a spacetime of class B, with covering space $(\mathcal{O}, g_{ab})$. □

Now, if a spacetime possesses a Killing vector field ξ^a, then any covering space of that spacetime possesses a corresponding Killing ξ'^a that projects to ξ^a. Consequently, if the existence of a Killing field is established for spacetimes of class B, it follows immediately from Proposition 3.1 that a corresponding Killing field exists for all spacetimes of class A. In particular, for analytic electrovac spacetimes of class B, Isenberg and Moncrief [11, 12] proved existence of a Killing field in a neighborhood of $\mathcal{N}$ which is normal to $\mathcal{N}$. Consequently, for any analytic, electrovac spacetime of class A, there also exists a Killing field in a neighborhood of $\mathcal{N}$ which is normal to $\mathcal{N}$. Thus, Hawking's theorem [1, 2] may be obtained as a corollary of the theorem of Isenberg and Moncrief together with Proposition 3.1.

The main aim of our paper is to extend the theorems of Hawking and of Isenberg and Moncrief to the smooth case. In view of the above remark, it suffices to extend the Isenberg-Moncrief theorem, since the extension of the Hawking theorem will then follow automatically. Thus, in the following, we shall restrict attention to spacetimes of class B.

For spacetimes of class B, $\mathcal{N}$ is a compact, orientable 3-manifold foliated by closed null geodesics. To discuss this situation we introduce some terminology (cf. [28] for more details). The "ordinary fibered solid torus" is defined as the set $D^2 \times S^1$ with the circles $\{p\} \times S^1$, $p \in D^2$, as "fibers". Here D^2 denotes the 2-dimensional closed unit disk. A "fibered solid torus" is obtained by cutting an

ordinary fibered solid torus along a disk $D^2 \times \{q\}$, for some $q \in S^1$, rotating one of the disks through an angle $\frac{m}{n}\, 2\,\pi$, where m, n are integers, and gluing them back again. While the central fiber now still closes after one cycle, the remaining fibers close in general only after n cycles. We note that there exists a fiber preserving $n : 1$ map $\widehat{\psi}$ of the ordinary fibered solid torus onto the fibered solid torus which is a local diffeomorphism that induces a $n : 1$ covering map on the central fiber.

As shown in [12], it follows from Epstein's theorem [29] that the null geodesics on $\mathcal{N}$ represent the fibers of a Seifert fibration. This means that any closed null geodesic has a "fibered neighborhood", i.e. a neighborhood fibered by closed null geodesics, which can be mapped by a fiber preserving diffeomorphism onto a fibered solid torus. Because $\mathcal{N}$ is compact, it can be covered by a finite number of such fibered neighborhoods, $\mathcal{N}_i$. For any neighborhood $\mathcal{N}_i$ there is an ordinary fibered solid torus $\widehat{\mathcal{N}}_i$ which is mapped by a fiber preserving map $\widehat{\psi}_i$ onto $\mathcal{N}_i$ as described above. Further we can choose a tubular spacetime neighborhood, $\mathcal{U}_i$, of $\mathcal{N}_i$, so that $\mathcal{U}_i$ has topology $D^2 \times \mathbb{R} \times S^1$ and the fibration of $\mathcal{N}_i$ extends to $\mathcal{U}_i$. There exists then a fibered extension $\widehat{\mathcal{U}}_i \simeq D^2 \times \mathbb{R} \times S^1$ of $\widehat{\mathcal{N}}_i$, with fibers $\{p\} \times S^1$, $p \in D^2 \times \mathbb{R}$, to which $\widehat{\psi}_i$ can be extended to a fiber preserving local diffeomorphism which maps $\widehat{\mathcal{U}}_i$ onto $\mathcal{U}_i$. We denote the extension again by $\widehat{\psi}_i$. Let $\mathcal{O}_i$ denote the universal covering space of $\widehat{\mathcal{U}}_i$ (so that $\mathcal{O}_i$ has topology $D^2 \times \mathbb{R}^2$). We denote the projection map from $\mathcal{O}_i$ onto $\widehat{\mathcal{U}}_i$ by $\widetilde{\psi}_i$, set $\psi_i = \widehat{\psi}_i \circ \widetilde{\psi}_i$, and denote the inverse image of $\mathcal{N}_i$ under ψ_i by $\widetilde{\mathcal{N}}_i$.

We will refer to $(\mathcal{O}_i, \psi_i^* g_{ab})$ as an *elementary spacetime region.* Note that for the case of a spacetime of class B constructed from a spacetime of class A in the manner of Proposition 3.1, $\mathcal{O}_i$ may be identified with a neighborhood of a portion of the horizon in the original (class A) spacetime.

Our main results will be based upon the following theorem, which may be extracted directly from the analysis of Isenberg and Moncrief [11, 12]:

Theorem 3.1 *(Moncrief & Isenberg) Let (M, g_{ab}) be a smooth electrovac spacetime of class B and let $(\mathcal{O}_i, \psi_i^* g_{ab})$ be an elementary spacetime region, as defined above. Then, there exists a Gaussian null coordinate system (u, r, x^3, x^4) (see Appendix A) covering a neighborhood, $\mathcal{O}'_i$, of $\widetilde{\mathcal{N}}_i$ in $\mathcal{O}_i$ so that the following properties hold (i) The coordinate range of u is $-\infty < u < \infty$ whereas the coordinate range of r is $-\epsilon < r < \epsilon$ for some $\epsilon > 0$, with the surface $r = 0$ being $\widetilde{\mathcal{N}}_i$. (ii) In $\mathcal{O}'_i$, the projection map $\widetilde{\psi}_i : \mathcal{O}_i \to \widehat{\mathcal{U}}_i$ is obtained by periodically identifying the coordinate u with some period $P \in \mathbb{R}$. Thus, in particular, the components of $\psi_i^* g_{ab}$ and $\psi_i^* F_{ab}$ in these coordinates are periodic functions of u with period P. (iii) We have, writing in the following for convenience g_{ab} and F_{ab} instead of $\psi_i^* g_{ab}$ and $\psi_i^* F_{ab}$,*

$$f|_{\widetilde{\mathcal{N}}_i} = -2\kappa_\circ \quad \text{and} \quad F_{uA}|_{\widetilde{\mathcal{N}}_i} = 0, \tag{3.1}$$

with $\kappa_\circ \in \mathbb{R}$, where f is defined in Appendix A. (iv) On $\widetilde{\mathcal{N}}_i$, the r-derivatives of the metric and Maxwell field tensor components up to any order are u-independent, i.e., in the notation of Appendix A

$$\frac{\partial}{\partial u}\left[\frac{\partial^n}{\partial r^n}\{f, h_A, g_{AB}; F_{ur}, F_{uA}, F_{rA}, F_{AB}\}\right]\Bigg|_{\widetilde{\mathcal{N}}_i} = 0, \tag{3.2}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Remark 3.1 *Along the null geodesic generators of $\widetilde{\mathcal{N}}_i$ the vector field $k^a = (\partial/\partial u)^a$ (which we take to be future directed) satisfies the equation*

$$k^a \nabla_a k^b = \kappa_\circ k^b. \tag{3.3}$$

Thus, if $\kappa_\circ > 0$, then all of the null geodesic generators of $\widetilde{\mathcal{N}}_i$ are past incomplete but future complete. If $\kappa_\circ < 0$ then all of the null geodesic generators of $\widetilde{\mathcal{N}}_i$ are past complete but future incomplete. Similarly, if $\kappa_\circ = 0$ (usually referred to as the "degenerate case"), then u is an affine parameter and all of the null geodesic generators of $\widetilde{\mathcal{N}}_i$ are complete in both the past and future directions.

Remark 3.2 *In the analytic case, eq.(3.2) directly implies that $k^a = (\partial/\partial u)^a$ is a Killing vector field in a neighborhood of $\widetilde{\mathcal{N}}_i$. Since the projection map $\widetilde{\psi}_i$ is obtained by periodically identifying the coordinate u, it follows immediately that k^a projects to a Killing vector field $\widehat{k}^a$ in a neighborhood of $\widehat{\mathcal{N}}_i$. Appealing then to the argument of [12], it can be shown that the $\widehat{k}^a$ further projects to a Killing vector field under the action of the map $\widehat{\psi}_i$, so that the map $\psi_i = \widetilde{\psi}_i \circ \widehat{\psi}_i$ projects $k^a = (\partial/\partial u)^a$ to a well-defined Killing field in a neighborhood of $\mathcal{N}_i$. The arguments of [11, 12] then establish that the local Killing fields obtained for each fibered neighborhood can be patched together to produce a global Killing field on a neighborhood of $\mathcal{N}$.*

In the next section, we shall generalize the result of Remark 3.2 to the smooth case. However, to do so we will need to impose the additional restriction that $\kappa_\circ \neq 0$, and we will prove existence only on a one-sided neighborhood of the horizon.

4 Existence of a Killing vector field

The main difficulty encounted when one attempts to generalize the Isenberg-Moncrief theorem to the smooth case is that suitable detailed information about the spacetime metric and Maxwell field is known only on $\mathcal{N}$ (see eq.(3.2) above). If a Killing field k^a exists, it is determined uniquely by the data k_a, $\nabla_{[a} k_{b]}$ at one point of $\mathcal{N}$, because equations (B.4), (B.6) imply a system of ODE's for the tetrad components k_j, $\nabla_{[i} k_{j]}$ along each C^1 curve. But the existence of a Killing field cannot be shown this way. Thus we will construct the Killing field as a solution to a PDE problem. However, $\mathcal{N}$ is a null surface, and thus, by itself, it does not comprise a suitable initial data surface for the relevant hyperbolic equations. We now remedy this difficulty by performing a suitable local extension of a neighborhood of $\widetilde{\mathcal{N}}_i$ which is covered by the Gaussian null coordinates of Theorem 3.1. This is achieved via the following Proposition:

Proposition 4.1 *Let $(\mathcal{O}_i, g_{ab}\,|_{\mathcal{O}_i})$ be an elementary spacetime region associated with an electrovac spacetime of class B such that $\kappa_\circ > 0$ (see eq.(3.1) above). Then, there exists an open neighborhood, $\mathcal{O}''_i$, of $\widetilde{\mathcal{N}}_i$ in $\mathcal{O}_i$ such that $(\mathcal{O}''_i, g_{ab}\,|_{\mathcal{O}''_i}, F_{ab}\,|_{\mathcal{O}''_i})$ can be extended to a smooth electrovac spacetime, $(\mathcal{O}^*, g^*_{ab}, F^*_{ab})$, that possesses a bifurcate null surface, $\widetilde{\mathcal{N}}^*$—i.e., $\widetilde{\mathcal{N}}^*$ is the union of two null hypersurfaces, $\mathcal{N}^*_1$ and $\mathcal{N}^*_2$, which intersect on a 2-dimensional spacelike surface, S—such that $\widetilde{\mathcal{N}}_i$ corresponds to the portion of $\mathcal{N}^*_1$ that lies to the future of S and $I^+[S] = \mathcal{O}''_i \cap I^+[\widetilde{\mathcal{N}}_i]$. Furthermore, the expansion and shear of both $\mathcal{N}^*_1$ and $\mathcal{N}^*_2$ vanish.*

Proof It follows from eq.(3.2) that in $\mathcal{O}'_i$, the spacetime metric g_{ab} can be decomposed as

$$g_{ab} = g^{(0)}_{ab} + \gamma_{ab} \tag{4.1}$$

where, in the Gaussian null coordinates of Theorem 3.1, the components, $g^{(0)}_{\mu\nu}$, of $g^{(0)}_{ab}$ are independent of u, whereas the components, $\gamma_{\mu\nu}$, of γ_{ab} and all of their derivatives with respect to r vanish at $r = 0$ (i.e., on $\widetilde{\mathcal{N}}_i$). Furthermore, taking account of the periodicity of $\gamma_{\mu\nu}$ in u (so that, in effect, the coordinates (u, x^3, x^4) have a compact range of variation), we see that for all integers $j \geq 0$ we have throughout $\mathcal{O}'_i$

$$|\gamma_{\mu\nu}| < C_j |r|^j \tag{4.2}$$

for some constants C_j. Similar relations hold for all partial derivatives of $\gamma_{\mu\nu}$.

It follows from eq.(4.2) that there is an open neighborhood of $\widetilde{\mathcal{N}}_i$ in the spacetime $(\mathcal{O}'_i, g^{(0)}_{ab})$ such that $g^{(0)}_{ab}$ defines a Lorentz metric. It is obvious that in this neighborhood, $\widetilde{\mathcal{N}}_i$ is a Killing horizon of $g^{(0)}_{ab}$ with respect to the Killing field $k^a = (\partial/\partial u)^a$. Consequently, by the results of [15, 16], we may

extend an open neighborhood, $\mathcal{O}''_i$, of $\widetilde{\mathcal{N}}_i$ in $\mathcal{O}_i$ to a smooth spacetime $(\mathcal{O}^*, g^{(0)*}_{ab})$, that possesses a bifurcate Killing horizon, $\widetilde{\mathcal{N}}^*$, with respect to $g^{(0)*}_{ab}$. Furthermore, with respect to the metric $g^{(0)*}_{ab}$, $\widetilde{\mathcal{N}}^*$ automatically satisfies all of the properties stated in the Proposition. In addition, by theorem 4.2 of [16], the extension can be chosen so that k^a extends to a Killing field, k^{*a}, of $g^{(0)*}_{ab}$ in $\mathcal{O}^*$, and $(\mathcal{O}^*, g^{(0)*}_{ab})$ possesses a "wedge reflection" isometry (see [16]); we assume that such a choice of extension has been made.

Let (u_0, r_0, x^3_0, x^4_0) denote the Gaussian null coordinates in $\mathcal{O}''_i$ associated with $g^{(0)}_{ab}$, such that on $\widetilde{\mathcal{N}}_i$ we have $r_0 = r = 0, u_0 = u, x^3_0 = x^3, x^4_0 = x^4$. Since γ_{ab} is smooth in $\mathcal{O}''_i$ and is periodic in u, it follows that each of the coordinates (u_0, r_0, x^3_0, x^4_0) are smooth functions of (u, r, x^3, x^4) which are periodic in u. It further follows that the Jacobian matrix of the transformation between (u_0, r_0, x^3_0, x^4_0) and (u, r, x^3, x^4) is uniformly bounded in $\mathcal{O}''_i$, and that, in addition, there exists a constant, c such that $|r| \le c|r_0|$ in $\mathcal{O}''_i$. Consequently, the components, $\gamma_{\mu_0\nu_0}$ of γ_{ab} in the Gaussian null coordinates associated with $g^{(0)}_{ab}$ satisfy for all integers $j \ge 0$

$$|\gamma_{\mu_0\nu_0}| < C'_j |r_0|^j \tag{4.3}$$

Let (U, V) denote the generalized Kruskal coordinates with respect to $g^{(0)}_{ab}$ introduced in [15, 16]. In terms of these coordinates, $\mathcal{O}''_i$ corresponds to the portion of $\mathcal{O}^*$ satisfying $U > 0$ and the wedge reflection isometry mentioned above is given by $U \to -U, V \to -V$. The null hypersurfaces $\mathcal{N}^*_1$ and $\mathcal{N}^*_2$ that comprise the bifurcate Killing horizon, $\widetilde{\mathcal{N}}^*$, of $g^{(0)*}_{ab}$ correspond to the hypersurfaces defined by $V = 0$ and $U = 0$, respectively.

It follows from eq.(23) of [15] that within $\mathcal{O}''_i$ we have

$$|r_0| < C|UV| \tag{4.4}$$

for some constant C. Hence, we obtain for all j

$$|\gamma_{\mu_0\nu_0}| < C''_j |UV|^j \tag{4.5}$$

with similar relations holding for all of the derivatives of $\gamma_{\mu_0\nu_0}$ with respect to the coordinates (u_0, r_0, x^3_0, x^4_0). Taking account of the transformation between the Gaussian null coordinates and the generalized Kruskal coordinates (see eqs. (24) and (25) of [15]), we see that the Kruskal components of γ_{ab} and all of their Kruskal coordinate derivatives also go to zero uniformly on compact subsets of (V, x^3, x^4) in the limit as $U \to 0$. It follows that the tensor field γ_{ab} on $\mathcal{O}''_i$ extends smoothly to $U = 0$—i.e., the null hypersurface $\mathcal{N}^*_2$ of $\mathcal{O}^*$—such that γ_{ab} and all of its derivatives vanish on $\mathcal{N}^*_2$. We now further extend γ_{ab} to the region $U < 0$—thereby defining a smooth tensor field γ^*_{ab} on all of $\mathcal{O}^*$—by requiring it to be invariant under the above wedge reflection isometry. In $\mathcal{O}^*$, we define

$$g^*_{ab} = g^{(0)*}_{ab} + \gamma^*_{ab} \tag{4.6}$$

Then g^*_{ab} is smooth in $\mathcal{O}^*$ and is invariant under the wedge reflection isometry. Furthermore, since γ^*_{ab} vanishes on $\widetilde{\mathcal{N}}^*$, it follows that $\widetilde{\mathcal{N}}^*$ is a bifurcate null surface with respect to g^*_{ab} and that k^{*a} is normal to $\widetilde{\mathcal{N}}^*$. In addition, on $\widetilde{\mathcal{N}}^*$ we have $\mathcal{L}_{k^*} g^{(0)*}_{ab} = 0$ (since k^{*a} is a Killing field of $g^{(0)*}_{ab}$) and $\mathcal{L}_{k^*} \gamma^*_{ab} = 0$ (since γ^*_{ab} and its derivatives vanish on $\widetilde{\mathcal{N}}^*$). Therefore, we have $\mathcal{L}_{k^*} g^*_{ab} = 0$ on $\widetilde{\mathcal{N}}^*$. By Lemma B.1, it follows that the expansion and shear of both $\mathcal{N}^*_1$ and $\mathcal{N}^*_2$ vanish.

Finally, by a similar construction (using the fact that $F_{uA}|_{\widetilde{\mathcal{N}}_i} = 0$; see eq.(3.1)), we can extend the Maxwell field F_{ab} in $\mathcal{O}''_i$ to a smooth Maxwell field F^*_{ab} in $\mathcal{O}^*$ which is invariant under the wedge reflection isometry. By hypothesis, (g^*_{ab}, F^*_{ab}) satisfies the Einstein-Maxwell equations in the region $U > 0$. By invariance under the wedge reflection isometry, (g^*_{ab}, F^*_{ab}) also satisfies the Einstein-Maxwell equations in the region $U < 0$. By continuity, the Einstein-Maxwell equations also are satisfied for $U = 0$, so (g^*_{ab}, F^*_{ab}) is a solution throughout $\mathcal{O}^*$. □

Remark 4.1 *By remark 3.1, the hypothesis that $\kappa_\circ > 0$ is equivalent to the condition that the null geodesic generators of $\widetilde{\mathcal{N}}_i$ are past incomplete. Therefore, it is clear that Proposition 4.1 also holds for $\kappa_\circ < 0$ if we interchange futures and pasts. However, no analog of Proposition 4.1 holds for the "degenerate case" $\kappa_\circ = 0$.*

We are now prepared to state and prove our main theorem:

Theorem 4.1 *Let (M, g_{ab}) be a smooth electrovac spacetime of class B for which the generators of the null hypersurface $\mathcal{N}$ are past incomplete. Then there exists an open neighborhood, $\mathcal{V}$ of $\mathcal{N}$ such that in $J^+[\mathcal{N}] \cap \mathcal{V}$ there exists a smooth Killing vector field k^a which is normal to $\mathcal{N}$. Furthermore, in $J^+[\mathcal{N}] \cap \mathcal{V}$ the electromagnetic field, F_{ab}, satisfies $\mathcal{L}_k F_{ab} = 0$.*

Proof As explained in Section 3, we can cover $\mathcal{N}$ by a finite number of fibered neighborhoods, $\mathcal{N}_i$. Let $\mathcal{O}_i$ denote the elementary spacetime region obtained by "unwrapping" a neighborhood of $\mathcal{N}_i$, as explained in Section 3. By remark 3.1, the past incompleteness of the null geodesic generators of $\mathcal{N}$ implies that $\kappa_\circ > 0$, so Proposition 4.1 holds. We now apply Proposition B.1 to the extended spacetime $\mathcal{O}^*$ to obtain existence of a Killing vector field in the domain of dependence of $\widetilde{\mathcal{N}}^* = \mathcal{N}_1^* \cup \mathcal{N}_2^*$. By restriction to $\mathcal{O}_i$, we thereby obtain a Killing field K^a (which also Lie derives the Maxwell field) on a one-sided neighborhood of $\widetilde{\mathcal{N}}_i$ of the form $J^+[\widetilde{\mathcal{N}}_i] \cap \widetilde{\mathcal{V}}_i$, where $\widetilde{\mathcal{V}}_i$ is an open neighborhood of $\widetilde{\mathcal{N}}_i$. Both K^a and $k^a = (\partial/\partial u)^a$ are tangent to the null geodesic generators of $\widetilde{\mathcal{N}}_i$, so on $\widetilde{\mathcal{N}}_i$ we clearly have $K^a = \varphi k^a$ for some function φ. Furthermore, on $\widetilde{\mathcal{N}}_i$, we have $\mathcal{L}_K g_{ab} = 0$ (since K^a is a Killing field) and $\mathcal{L}_k g_{ab} = 0$ (as noted in the proof of Proposition 4.1). It follows immediately that $\nabla_a \varphi = 0$ on $\widetilde{\mathcal{N}}_i$, so we may rescale K^a so that $K^a = k^a$ on $\widetilde{\mathcal{N}}_i$. Since the construction of k^a off of $\widetilde{\mathcal{N}}_i$ (as described in Appendix A) is identical to that which must be satisfied by a Killing field (as described in Remark B.1 below), it follows that $K^a = k^a$ within their common domain of definition. Thus, the vector field $k^a = (\partial/\partial u)^a$—which previously had been shown to be a Killing field in the analytic case—also is a Killing field in the smooth case in $J^+[\widetilde{\mathcal{N}}_i] \cap \widetilde{\mathcal{V}}_i$. By exactly the same arguments as given in [11, 12] (see Remark 3.2 above) it then follows that the map $\psi_i = \widetilde{\psi}_i \circ \widehat{\psi}_i$ projects $k^a = (\partial/\partial u)^a$ to a well-defined Killing field in a one-sided neighborhood of $\mathcal{N}_i$, and that the local Killing fields obtained for each fibered neighborhood can be patched together to produce a global Killing field on a one-sided neighborhood of $\mathcal{N}$ of the form $J^+[\mathcal{N}] \cap \mathcal{V}$ where $\mathcal{V} = \cup_i \psi_i[\mathcal{V}_i]$. □

In view of Proposition 3.1, we have the following Corollary

Corollary 4.1 *Let (M, g_{ab}) be a smooth electrovac spacetime of class A for which the generators of the event horizon $\mathcal{N}$ are past incomplete. Then there exists an open neighborhood, $\mathcal{V}$ of $\mathcal{N}$ such that in $J^+[\mathcal{N}] \cap \mathcal{V}$ there exists a smooth Killing vector field k^a which is normal to $\mathcal{N}$. Furthermore, in $J^+[\mathcal{N}] \cap \mathcal{V}$ the electromagnetic field, F_{ab}, satisfies $\mathcal{L}_k F_{ab} = 0$.*

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A Gaussian null coordinate systems

In this Appendix the construction of a local Gaussian null coordinate systems will be recalled.

Let (M, g_{ab}) be a spacetime, let $\mathcal{N}$ be a smooth null hypersurface, and let ς be a smooth spacelike 2-surface lying in $\mathcal{N}$. Let (x^3, x^4) be coordinates on an open subset $\widetilde{\varsigma}$ of ς. On a neighborhood of $\widetilde{\varsigma}$ in $\mathcal{N}$, let k^a be a smooth, non-vanishing normal vector field to $\mathcal{N}$, so that the integral curves of k^a are the null geodesic generators of $\mathcal{N}$. Without loss of generality, we may assume that k^a is future directed.

On a sufficiently small open neighborhood, S, of $\widetilde{\varsigma} \times \{0\}$ in $\widetilde{\varsigma} \times \mathbb{R}$, let $\psi : S \to \mathcal{N}$ be the map which takes (q, u) into the point of $\mathcal{N}$ lying at parameter value u along the integral curve of k^a starting at q. Then, ψ is C^∞, and it follows from the inverse function theorem that ψ is $1:1$ and onto from an open neighborhood of $\widetilde{\varsigma} \times \{0\}$ onto an open neighborhood, $\widetilde{\mathcal{N}}$, of $\widetilde{\varsigma}$ in $\mathcal{N}$. Extend the functions x^3, x^4 from $\widetilde{\varsigma}$ onto $\widetilde{\mathcal{N}}$ by keeping their values constant along the integral curves of k^a. Then (u, x^3, x^4) are coordinates on $\widetilde{\mathcal{N}}$.

At each $p \in \widetilde{\mathcal{N}}$ let l^a be the unique null vector field on $\widetilde{\mathcal{N}}$ satisfying $l^a k_a = 1$ and $l^a X_a = 0$ for all X^a which are tangent to $\widetilde{\mathcal{N}}$ and satisfy $X^a \nabla_a u = 0$. On a sufficiently small open neighborhood, Q, of $\widetilde{\mathcal{N}} \times \{0\}$ in $\widetilde{\mathcal{N}} \times \mathbb{R}$, let $\Psi : Q \to M$ be the map which takes $(p, r) \in Q$ into the point of M lying at affine parameter value r along the null geodesic starting at p with tangent l^a. Then Ψ is C^∞ and it follows from the inverse function theorem that Ψ is $1:1$ and onto from an open neighborhood of $\widetilde{\mathcal{N}} \times \{0\}$ onto an open neighborhood, $\mathcal{O}$, of $\widetilde{\mathcal{N}}$ in M. We extend the functions u, x^3, x^4 from $\widetilde{\mathcal{N}}$ to $\mathcal{O}$ by requiring their values to be constant along each null geodesic determined by l^a. Then (u, r, x^3, x^4) yields a coordinate system on $\mathcal{O}$ which will be referred to as *Gaussian null* coordinate system. Note that on $\widetilde{\mathcal{N}}$ we have $k^a = (\partial/\partial u)^a$.

Since by construction the vector field $l^a = (\partial/\partial r)^a$ is everywhere tangent to null geodesics we have that $g_{rr} = 0$ throughout $\mathcal{O}$. Furthermore, we have that the metric functions g_{ru}, g_{r3}, g_{r4} are independent of r, i.e. $g_{ru} = 1, g_{r3} = g_{r4} = 0$ throughout $\mathcal{O}$. In addition, as a direct consequence of the above construction, g_{uu} and g_{uA} vanish on $\widetilde{\mathcal{N}}$. Hence, within $\mathcal{O}$, there exist smooth functions f and h_A, with $f|_{\widetilde{\mathcal{N}}} = (\partial g_{uu}/\partial r)\,|_{r=0}$ and $h_A|_{\widetilde{\mathcal{N}}} = (\partial g_{uA}/\partial r)\,|_{r=0}$, so that the spacetime metric in $\mathcal{O}$ takes the form

$$\mathrm{d}s^2 = r \cdot f \mathrm{d}u^2 + 2\mathrm{d}r\mathrm{d}u + 2r \cdot h_A \mathrm{d}u\mathrm{d}x^A + g_{AB}\mathrm{d}x^A\mathrm{d}x^B \tag{A.1}$$

where g_{AB} are smooth functions of u, r, x^3, x^4 in $\mathcal{O}$ such that g_{AB} is a negative definite 2×2 matrix, and the uppercase Latin indices take the values $3, 4$.

B The existence of a Killing field tangent to the horizon

The purpose of this section is to prove the following fact.

Proposition B.1 *Suppose that (M, g, F) is a time oriented solution to the Einstein-Maxwell equations with Maxwell field F (without sources). Let $\mathcal{N}_1$, $\mathcal{N}_2$ be smooth null hypersurfaces with connected space-like boundary $\mathcal{Z}$, smoothly embedded in M, which are generated by the future directed null geodesics orthogonal to $\mathcal{Z}$. Assume that $\mathcal{N}_1 \cup \mathcal{N}_2$ is achronal. Then there exists on the future domain of dependence, D^+, of $\mathcal{N}_1 \cup \mathcal{N}_2$ a non-trivial Killing field K which is tangent to the null generators of $\mathcal{N}_1$ and $\mathcal{N}_2$ if and only if these null hypersurfaces are expansion and shear free. If it exists, the Killing field is unique up to a constant factor and we have $\mathcal{L}_K F = 0$.*

Remark B.1 *We shall prove Proposition B.1 by first deducing the form that K must have on $\mathcal{N}_1 \cup \mathcal{N}_2$, then defining K on D^+ by evolution from $\mathcal{N}_1 \cup \mathcal{N}_2$ of a wave equation that must be satisfied*

by a Killing field, and, finally, proving that the resulting K is indeed a Killing field. An alternative, more geometric, approach to the proof of Proposition B.1 would be to proceed as follows. Suppose that the Killing field K of Proposition B.1 exists. Denote by ψ_t the local 1-parameter group of isometries associated with K. Since ψ_t maps $\mathcal{N}_1 \cup \mathcal{N}_2$ into itself, geodesics into geodesics, and preserves affine parameterization, we can describe for given t the action of ψ_t on a neighborhood of $\mathcal{N}_1 \cup \mathcal{N}_2$ in D^+ in terms of its action on geodesics passing through $\mathcal{N}_1 \cup \mathcal{N}_2$ into D^+ and affine parameters which vanish on $\mathcal{N}_1 \cup \mathcal{N}_2$. There are various possibilities, one could employ e.g. the future directed time-like geodesics starting on $\mathcal{Z}$ or the null geodesics which generate double null coordinates adapted to $\mathcal{N}_1 \cup \mathcal{N}_2$. Changing the point of view, one could try to use such a description to define maps ψ_t, show that they define a local group of isometries, and then define K as the corresponding Killing field. To show that $\psi_t^* g = g$, one would prove this relation on $\mathcal{N}_1 \cup \mathcal{N}_2$ and then invoke the uniqueness for the characteristic initial value problem for the Einstein-Maxwell equations to show that this relation holds also in a neighborhood of $\mathcal{N}_1 \cup \mathcal{N}_2$ in D^+. For this to work we would need to show that $\psi_t^* g$ has a certain smoothness (C^2 say). This is not so difficult away from $\mathcal{N}_1 \cup \mathcal{N}_2$ but it is delicate near the initial hypersurface. The discussion would need to take into account the properties of the underlying space-time exhibited in Lemmas B.1, B.2 below and would become quite tedious. For this reason we have chosen not to proceed in this manner.

It will be convenient to use the formalism, notation, and conventions of [30] in a gauge adapted to our geometrical situation. Since we will be using the tetrad formalism, throughout this Appendix we shall omit all abstract indices $a, b, c...$ on tensors and use the indices $i, j, k...$ to denote the components of tensors in our tetrad. We begin by choosing smooth coordinates $u = x^1$, $r = x^2$, x^A, $A = 3, 4$, and a smooth tetrad field

$$z_1 = l,\ z_2 = n\ z_3 = m,\ z_4 = \overline{m},$$

with $g_{ik} = g(z_i, z_k)$ such that $g_{12} = g_{21} = 1$, $g_{34} = g_{43} = -1$ are the only non-vanishing scalar products. Let x^A be coordinates on a connected open subset ζ of $\mathcal{Z}$ on which also m, $\overline{m}$ can be introduced such that they are tangent to ζ. On ζ we set $x^1 = 0$, $x^2 = 0$ and assume that l is tangent to $\mathcal{N}_1$, n is tangent to $\mathcal{N}_2$, and both are future directed. The possible choices which can be made above will represent the remaining freedom in our gauge. We assume

$$\nabla_{z_2} z_2 = 0, \quad < z_2, d\,x^\mu >= \delta^\mu{}_1, \quad \text{on} \quad N_2',$$

and set $\zeta_c = \{x^1 = c\} \subset \mathcal{N}_2'$ for $c \geq 0$, where $\mathcal{N}_2'$ denotes the subset of $\mathcal{N}_2$ generated by the null geodesics starting on ζ. We assume that m, $\overline{m}$ are tangent to ζ_c. From the transformation law of the spin coefficient γ under rotations $m \to e^{i\,\phi}\,m$ we find that we can always assume that $\gamma = \overline{\gamma}$ on $\mathcal{N}_2'$. With this assumption m, whence also l, will be fixed uniquely on $\mathcal{N}_2'$ and we have

$$\gamma = 0, \quad \nu = 0 \quad \text{on} \quad \mathcal{N}_2'.$$

The coordinates and the frame are extended off $\mathcal{N}_2'$ such that

$$\nabla_{z_1} z_i = 0, \quad < z_1, d\,x^\mu >= \delta^\mu{}_2.$$

On a certain neighborhood, D, of $\mathcal{N}_1' \cup \mathcal{N}_2'$ in D^+ (where $\mathcal{N}_1'$ is the subset of $\mathcal{N}_1$ generated by the null geodesics starting on ζ), we obtain by this procedure a smooth coordinate system and a smooth frame field which has in these coordinates the local expression

$$l_\mu = \delta^1{}_\mu, \quad l^\mu = \delta^\mu{}_2, \quad n^\mu = \delta^\mu{}_1 + U\,\delta^\mu{}_2 + X^A\,\delta^\mu{}_A, \quad m^\mu = \omega\,\delta^\mu{}_2 + \xi^A\,\delta^\mu{}_A.$$

We have $\mathcal{N}_1' = \{x^1 = u = 0\}$, $\mathcal{N}_2' = \{x^2 = r = 0\}$ and

$$\kappa = 0, \quad \epsilon = 0, \quad \pi = 0, \quad \tau = \overline{\alpha} + \beta \quad \text{on} \quad D, \quad U = 0, \quad X^A = 0, \quad \omega = 0 \quad \text{on} \quad \mathcal{N}_2'.$$

We shall use alternatively the Ricci rotation coefficients defined by $\nabla_i z_j \equiv \nabla_{z_i} z_j = \gamma_j{}^l{}_i z_l$ or their representation in terms of spin coefficients as given in [30]. The gauge above will be used in many local considerations whose results extend immediately to all of $\mathcal{N}_1$ and $\mathcal{N}_2$. We shall then always state the extended result.

We begin by showing the necessity of the conditions on the null hypersurfaces in Proposition B.1 and some of their consequences.

Lemma B.1 *Let $\mathcal{N}$ be a smooth null hypersurface of the space-time (M, g) and X a smooth vector field on M which is tangent to the null generators of $\mathcal{N}$ and does not vanish there. If g' denotes the pull back of g to $\mathcal{N}$, then $\mathcal{L}_X g' = 0$ on $\mathcal{N}$ if and only if the null generators of $\mathcal{N}$ are expansion and shear free.*

Proof We can assume that $\mathcal{N}$ coincides with the hypersurface $\mathcal{N}_1$. Then we have in our gauge $X = X^1 z_1$ on $\mathcal{N}$ with $X^1 \neq 0$, and $\mathcal{L}_X g' = 0$ translates into $0 = \nabla_{(i} X_{j)} = z_{(i}(X^1)\,\delta^2{}_{j)} - \gamma_{(j}{}^2{}_{i)} X^1$ on $\mathcal{N}$ with $i, j \neq 2$. By our gauge this is equivalent to $0 = \gamma_{(A}{}^2{}_{B)} = -\sigma\,\delta^3{}_A\,\delta^3{}_B - Re\,\rho\,\delta^3{}_{(A}\,\delta^4{}_{B)} - \bar{\sigma}\,\delta^4{}_A\,\delta^4{}_B$. Since ρ is real on the hypersurface $\mathcal{N}$, the assertion follows. $\square$

Lemma B.2 *If the null hypersurfaces $\mathcal{N}_1$, $\mathcal{N}_2$ are expansion and shear free, then the frame coefficients, the spin coefficients, the components Ψ_i of the conformal Weyl spinor field, the components ϕ_k of the Maxwell spinor field, and the components $\Phi_{ik} = k\,\phi_i\,\bar{\phi}_k$ of the Ricci spinor field are uniquely determined in our gauge on $\mathcal{N}_1'$ and $\mathcal{N}_2'$ by the field equations and the data*

$$\phi_1, \quad \tau, \quad \xi^A, \;\; A = 3, 4, \quad \text{on} \quad \mathcal{Z}. \tag{B.1}$$

In particular, we have

$$\Psi_0 = 0,\;\; \Psi_1 = 0,\;\; \phi_0 = 0,\;\; \Phi_{0k} = \overline{\Phi}_{k0} = 0,$$
$$D\,\phi_1 = 0,\;\; D\,\phi_2 = \overline{\delta}\,\phi_1,\;\; \phi_2 = r\,\overline{\delta}\,\phi_1,\;\; \omega = -r\,\tau,\;\; \mu = r\,\Psi_2, \quad \text{on} \quad \mathcal{N}_1', \tag{B.2}$$

$$\Psi_4 = 0,\;\; \Psi_3 = 0,\;\; \phi_2 = 0,\;\; \Phi_{i2} = \overline{\Phi}_{2i} = 0,$$
$$\Delta\,\phi_1 = 0,\;\; \Delta\,\phi_0 = \delta\,\phi_1 - 2\,\tau\,\phi_1,\;\; \phi_0 = u\,(\delta\,\phi_1 - 2\,\tau\,\phi_1),\;\; \rho = u\,(\overline{\delta}\,\tau - 2\,\alpha\,\tau - \Psi_2) \quad \text{on} \quad \mathcal{N}_2'. \tag{B.3}$$

Proof In our gauge the relations $\Psi_0 = 0$, $\Phi_{00} = k\,\phi_0\,\overline{\phi}_0 = 0$ on $\mathcal{N}_1'$ are an immediate consequence of the NP equations and our assumption that $\rho = 0$, $\sigma = 0$ on $\mathcal{N}_1'$. Similarly, the assumptions $\mu = 0$, $\lambda = 0$ on $\mathcal{N}_2'$ imply $\Psi_4 = 0$, $\Phi_{22} = k\,\phi_2\,\overline{\phi}_2 = 0$ on $\mathcal{N}_2'$. The relation $\Phi_{ij} = k\,\phi_i\,\overline{\phi}_j$ implies the other statements on the Ricci spinor on $\mathcal{N}_1'$, $\mathcal{N}_2'$ and it allows us to determine Φ_{11} on $\mathcal{Z}$ from the data (B.1).

The NP equations involving only the operators δ, $\overline{\delta}$, the data (B.1), and our gauge conditions allow us to calculate the functions α, β, $\Psi_1 = 0$, Ψ_2, $\Psi_3 = 0$ on $\mathcal{Z}$. Then all metric coefficients, spin coefficients, and the Weyl, Ricci, and Maxwell spinor fields are known on $\mathcal{Z}$. The remaining assertions follow by integrating in the appropriate order the NP equations (cf. also the appendix of [30]) involving the operator D on $\mathcal{N}_1'$ and the equations involving the operator Δ on $\mathcal{N}_2'$. $\square$

Lemma B.3 *A Killing field K as considered in Proposition B.1 satisfies, up to a constant factor,*

$$K = r\,z_1 \quad \text{on} \quad \mathcal{N}_1, \quad K = -u\,z_2 \quad \text{on} \quad \mathcal{N}_2.$$

Proof We note, first, that the statement above is reasonable, because the vector fields z_1, z_2 can be defined globally on $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively and the form of K given above is preserved under rescalings consistent which our gauge freedom on $\mathcal{Z}$.

Writing $K = K^i\, z_i$, we have by our assumptions $K = K^1\, z_1$ on $\mathcal{N}_1$, $K = K^2\, z_2$ on $\mathcal{N}_2$ for some smooth functions K^1, K^2 which vanish on $\mathcal{Z}$. To determine their explicit form we use, in addition to the Killing equation

$$\mathcal{L}_K\, g_{ij} = \nabla_i\, K_j + \nabla_j\, K_i = 0 \tag{B.4}$$

the identity

$$\nabla_i\, \nabla_j\, K_l + K_m\, R^m{}_{ilj} = \frac{1}{2}\, \{\nabla_i(\mathcal{L}_K\, g_{lj}) + \nabla_j(\mathcal{L}_K\, g_{li}) - \nabla_l(\mathcal{L}_K\, g_{ij})\} \tag{B.5}$$

which holds for arbitrary smooth vector field K and metric g and which implies, together with eq.(B.4), the integrability condition

$$\nabla_i\, \nabla_j\, K_l + K_m\, R^m{}_{ilj} = 0. \tag{B.6}$$

The restriction of eq.(B.4) to ζ gives $\nabla_i\, K_j = 2\, h\, \delta^1{}_{[i}\, \delta^2{}_{j]}$ with $h = z_1\,(K^1) = -z_2\,(K^2)$. Using this expression to evaluate eq.(B.6) on ζ for $i = A = 3, 4$, and observing that ζ is connected we get $z_A(h) = 0$, whence $h = const.$ on ζ. Since $\mathcal{Z}$ is connected the same expression for $\nabla_i\, K_j$ will be obtained everywhere on $\mathcal{Z}$ with the same constant h. If h were zero, K would vanish identically by equations (B.4) and (B.6). Since K is assumed to be non-trivial we have $h \neq 0$ and can rescale K to achieve $h = 1$.

Equations (B.4), (B.6) imply in our gauge

$$z_1\,(K_2) = -\nabla_2\, K_1, \quad z_1\,(\nabla_2\, K_1) = 0 \quad \text{on} \quad \mathcal{N}_1',$$

$$z_2\,(K_1) = -\nabla_1\, K_2, \quad z_2\,(\nabla_1\, K_2) = 0 \quad \text{on} \quad \mathcal{N}_2',$$

which, together with the value of $\nabla_i\, K_j$, $i, j = 1, 2$, on $\mathcal{Z}$ entail our assertion. $\square$

Taking into account $\rho = 0$, $\sigma = 0$ on $\mathcal{N}_1$, $\mu = 0$, $\lambda = 0$ on $\mathcal{N}_2$, and in particular eq.(B.2), we immediately get the following.

Lemma B.4 *By calculations which involve only inner derivatives on the respective null hypersurface one obtains from eq.(B.3)*

$$\nabla_i\, K_j = \delta^1{}_i\, \delta^2{}_j, \quad i \neq 2, \quad \text{on} \quad \mathcal{N}_1, \tag{B.7}$$

$$\nabla_i\, K_j = -\delta^2{}_i\, \delta^1{}_j + u\, \tau\, \delta^3{}_i\, \delta^1{}_j + u\, \overline{\tau}\, \delta^4{}_i\, \delta^1{}_j, \quad i \neq 1, \quad \text{on} \quad \mathcal{N}_2. \tag{B.8}$$

Equation (B.6) implies the hyperbolic system

$$\nabla_i\, \nabla^i\, K_l - K_m\, R^m{}_l = 0, \tag{B.9}$$

and the initial data for the Killing field we wish to construct are given by Lemma B.3. Both have an invariant meaning.

Lemma B.5 *There exists a unique smooth solution, K, of eq.(B.9) on D^+ which takes on $\mathcal{N}_1 \cup \mathcal{N}_2$ the values given in Lemma B.3.*

Proof The uniqueness of the solution is an immediate consequence of standard energy estimates. The results in [31] or [32] entail the existence of a unique smooth solution of eq.(B.9) for the data given in Lemma B.3 on an open neighborhood of ζ in $D \cap D^+(\mathcal{N}_1' \cup \mathcal{N}_2')$. These local solutions can be patched together to yield a solution in some neighborhood of $\mathcal{Z}$ in D. Because of the linearity of eq.(B.9) this solution can be extended (e.g. by a patching procedure) to all of D^+. $\square$

Lemma B.6 *The vector field K of Lemma B.5 satisfies $\mathcal{L}_K\, g = 0$ and $\mathcal{L}_K\, F = 0$ on D^+.*

Proof The equations above need to be deduced from the structure of the data in Lemma B.3 and from eq.(B.9). Applying ∇_j to (B.9) and commuting derivatives we get

$$\nabla_i \nabla^i (\mathcal{L}_K g_{jl}) = 2 \mathcal{L}_K R_{jl} + 2 R^i{}_{jl}{}^k (\mathcal{L}_K g_{ik}) - 2 R^i{}_{(j} (\mathcal{L}_K g_{l)i}). \tag{B.10}$$

The Einstein equations give

$$\mathcal{L}_K R_{ij} = k' \left\{ 2 F_{(i}{}^l (\mathcal{L}_K F_{j)l}) - \frac{1}{2} g_{ij} (\mathcal{L}_K F_{kl}) F^{kl} - (\mathcal{L}_K g_{kl}) F_i{}^k F_j{}^l \right. \tag{B.11}$$

$$\left. - \frac{1}{4} (\mathcal{L}_K g_{ij}) F_{kl} F^{kl} + \frac{1}{2} g_{ij} (\mathcal{L}_K g_{kl}) F^{km} F^l{}_m \right\}.$$

The identity $d \mathcal{L}_K F = d (i_K d F + d i_K F)$ together with Maxwell's equations implies

$$\nabla_{[i} (\mathcal{L}_K F_{jl]}) = 0. \tag{B.12}$$

Applying $\mathcal{L}_K$ to the second part of Maxwell's equations and using the identity (B.5) as well as the fact that K solves eq.(B.9), we get

$$\nabla^i (\mathcal{L}_K F_{ik}) = F^{jl} \nabla_j (\mathcal{L}_K g_{lk}) + (\mathcal{L}_K g_{jl}) \nabla^j F^l{}_k. \tag{B.13}$$

Substituting eq.(B.11) in eq.(B.10), we can view the system (B.10), (B.12), (B.13) as a homogeneous linear system for the unknowns $\mathcal{L}_K g$, $\mathcal{L}_K F$. This system implies a linear symmetric hyperbolic system for the unknowns $\mathcal{L}_K g$, $\nabla \mathcal{L}_K g$, $\mathcal{L}_K F$ (cf. [33]).

We shall show now that these unknowns vanish on $\mathcal{N}_1 \cup \mathcal{N}_2$. The standard energy estimates for symmetric hyperbolic systems then imply that the fields vanish in fact on D^+, which will prove our lemma and thus Proposition B.1. Equation (B.9) restricted to $\mathcal{N}_1'$ reads

$$0 = \nabla_i \nabla^i K_l - K_m R^m{}_l = 2 (\nabla_1 \nabla_2 K_l - \nabla_3 \nabla_4 K_l) - K_m (R^m{}_l + R^m{}_{l21} - R^m{}_{l43}).$$

Using this equation together with eqs.(B.2) and (B.7) and our gauge conditions, we obtain by a direct calculation a system of ODE's of the form

$$\nabla_1 (\nabla_{(i} K_{j)}) = H_{ij} (\nabla_{(k} K_{l)}),$$

on the null generators of $\mathcal{N}_1'$. Here H_{ij} is a linear function of the indicated argument (suppressing the dependence on the points of $\mathcal{N}_1'$). Since $\nabla_{(k} K_{l)} = 0$ on $\mathcal{Z}$, we conclude that $\mathcal{L}_K g = 0$ on $\mathcal{N}_1$. An analogous argument involving eqs.(B.3) and (B.8) shows that $\mathcal{L}_K g = 0$ on $\mathcal{N}_2$. It follows in particular that $\nabla \mathcal{L}_K g = 0$ on $\mathcal{Z}$.

Writing $(\mathcal{L}_K F)_{AA'BB'} = \epsilon_{A'B'} p_{AB} + \epsilon_{AB} \overline{p}_{A'B'}$ and using (B.3) we find on $\mathcal{N}_1'$ in NP notation

$$p_0 = r D \phi_0 + \phi_0, \quad p_1 = r D \phi_1, \quad p_2 = r D \phi_2 - \phi_2.$$

It follows from eq.(B.2) that p_{AB}, whence $\mathcal{L}_K F$, vanishes on $\mathcal{N}_1$. An analogous argument involving eq.(B.3) shows that $\mathcal{L}_K F$ vanishes on $\mathcal{N}_2$.

Observing in eqs.(B.11) and (B.10) that $\mathcal{L}_K F = 0$, $\mathcal{L}_K g = 0$, whence also $\nabla_l (\nabla_{(i} K_{j)}) = 0$ for $l \neq 2$ on $\mathcal{N}_1'$, we obtain there

$$0 = \nabla_k \nabla^k (\nabla_{(i} K_{j)}) = 2 \nabla_1 (\nabla_2 (\nabla_{(i} K_{j)})).$$

We conclude that $\nabla \mathcal{L}_K g$, which vanishes on $\mathcal{Z}$, vanishes on $\mathcal{N}_1$. In a similar way it follows that $\nabla \mathcal{L}_K g = 0$ on $\mathcal{N}_2$. This completes the proof. $\square$

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